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On the Fourier–Gauss transforms of some q-exponential and q-trigonometric functions

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Abstract. We examine the properties of q-exponential and q-trigonometric functions, recently introduced and discussed in the literature. It is shown that they are related to Jackson's q-analogues of the exponential and trigonometric functions by classical Fourier–Gauss transformations.

The exponential function e^z is known to possess the most remarkable properties of the elementary functions of classical analysis. It represents an entire function in the complex z plane, e^z is its own derivative, and it obeys the addition law $e^{z_1+z_2} = e^{z_1}e^{z_2}$. Because of the simple analytic behaviour of the exponential function, it is very frequently used in various branches of mathematics. For our purposes it is sufficient to mention just two instances of such applications: the exponential function serves as a kernel for the Fourier and Laplace integral transforms and it forms a basis for constructing an exponential mapping from Lie algebras to Lie groups.

To have the complete theory of q-special functions [1–3] it is thus very important to determine an appropriate q-extension of the exponential function. This problem has recently attracted much attention in the literature [4–10]. The goal of this short paper is to examine the properties of q-exponential and q-trigonometric functions, discussed in [4–10]. It is shown that they are related by classical Fourier–Gauss transformations to the q-analogues of exponential and trigonometric functions, introduced earlier by Jackson [11].

We start with a two-parameter q-exponential function, defined [5] as

$$\mathcal{E}_q^{IZ}(x; a, b) := \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} \left(aq^{(1-n)/2}\,e^{i\theta},\, aq^{(1-n)/2}e^{-i\theta};\, q\right)_n b^n \tag{1}$$

where $x = \cos\theta$, $(a;q)_0 = 1$ and $(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$, $n = 1, 2, 3, \ldots$, is the q-shifted factorial with the convention $(a_1, \ldots, a_k; q)_n = \prod_{j=1}^{k}(a_j;q)_n$. Throughout this paper we will employ the standard notations of q-special functions [2]. The q-exponential function (1) is also expressible as a sum of two ${}_4\phi_3$ basic hypergeometric series, i.e.

$$\mathcal{E}_q^{IZ}(x; a, b) = {}_4\phi_3\left(\begin{matrix} aq^{1/2}e^{i\theta},\, aq^{1/2}e^{-i\theta},\, q^{1/2}e^{i\theta}/a,\, q^{1/2}e^{-i\theta}/a \\ q^{1/2},\, -q^{1/2},\, -q \end{matrix};\, q,\, a^2b^2\right)$$
$$+\frac{bq^{1/4}}{1-q}\,(1-2ax+a^2)\,{}_4\phi_3\left(\begin{matrix} aqe^{i\theta},\, aqe^{-i\theta},\, qe^{i\theta}/a,\, qe^{-i\theta}/a \\ q^{3/2},\, -q^{3/2},\, -q \end{matrix};\, q,\, a^2b^2\right). \tag{2}$$

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An important advance in the study of properties of the q-exponential function (1) was achieved by Suslov, who proposed considering it as a function of two independent variables $x = \cos\theta$ and $y = \cos\varphi$ and a parameter α by replacing $a = -\mathrm{e}^{\mathrm{i}\varphi}$ and $b = \alpha\mathrm{e}^{-\mathrm{i}\varphi}$ in (1) [10]. Then the function

$$\begin{aligned}\mathcal{E}_q(x, y; \alpha) &:= e_{q^2}(q\alpha^2)\, E_{q^2}(-\alpha^2)\, \mathcal{E}_q^{IZ}(x; -\mathrm{e}^{\mathrm{i}\varphi}, \alpha\mathrm{e}^{-\mathrm{i}\varphi}) \\ &= e_{q^2}(q\alpha^2)\, E_{q^2}(-\alpha^2) \\ &\quad\times \sum_{n=0}^{\infty} \frac{q^{n^2/4}}{(q;q)_n} \left(-q^{(1-n)/2}\mathrm{e}^{\mathrm{i}(\theta+\varphi)}, -q^{(1-n)/2}\mathrm{e}^{\mathrm{i}(\varphi-\theta)}; q\right)_n (\alpha\mathrm{e}^{-\mathrm{i}\varphi})^n \end{aligned} \tag{3}$$

is symmetric in the variables x and y. Moreover, these variables are separated, i.e.

$$\mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(x, 0; \alpha)\, \mathcal{E}_q(0, y; \alpha)\,. \tag{4}$$

Note that the normalization constant $e_{q^2}(q\alpha^2)\, E_{q^2}(-\alpha^2)$ in (3) is chosen in such a way that $\mathcal{E}_q(0, 0; \alpha) = 1$ [10]; $e_q(z)$ and $E_q(z)$ are Jackson's q-exponential functions [11]

$$e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n} = (z;q)_\infty^{-1} \qquad E_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{(q;q)_n} z^n = (-z;q)_\infty\,. \tag{5}$$

As a q-analogue of the addition law for the exponential function e^z, the relation (4) reveals the importance of the particular case of (3), given explicity by

$$\begin{aligned}\mathcal{E}_q(x, 0; \alpha) = \mathcal{E}_q(x; \alpha) &= e_{q^2}(q\alpha^2)\, E_{q^2}(-\alpha^2)\, \mathcal{E}_q^{IZ}(x; -\mathrm{i}, -\mathrm{i}\alpha) \\ &= e_{q^2}(q\alpha^2)\, E_{q^2}(-\alpha^2) \Big[{}_2\phi_1(-q\mathrm{e}^{2\mathrm{i}\theta}, -q\mathrm{e}^{-2\mathrm{i}\theta}; q; q^2, \alpha^2) \\ &\quad + \frac{2q^{1/4}\alpha}{1-q}\, x\, {}_2\phi_1(-q^2\mathrm{e}^{2\mathrm{i}\theta}, -q^2\mathrm{e}^{-2\mathrm{i}\theta}; q^3; q^2, \alpha^2) \Big]. \end{aligned} \tag{6}$$

Once the relation between $\mathcal{E}_q(x; \alpha)$ and $\mathcal{E}_q^{IZ}(x; -\mathrm{i}, -\mathrm{i}\alpha)$ is clear, one can use the known properties of the latter [5, 9]. Firstly, $\mathcal{E}_q^{IZ}(x; -\mathrm{i}, -\mathrm{i}\alpha)$ can also be considered as a generating function for the continuous q-Hermite polynomials [5]. Therefore, as follows from (6),

$$\mathcal{E}_q(x; \alpha) = e_{q^2}(q\alpha^2) \sum_{n=0}^{\infty} \frac{q^{n^2/4}\alpha^n}{(q;q)_n} H_n(x|q)\,. \tag{7}$$

Secondly, relation (7) defines the Fourier–Gauss transformation properties of $\mathcal{E}_q(x; \alpha)$. Indeed, since the continuous q-Hermite polynomials $H_n(x|q)$ and q^{-1}-Hermite polynomials $h_n(x|q) := \mathrm{i}^{-n} H_n(\mathrm{i}x|q^{-1})$ are related to each other by the Fourier–Gauss transform [12]:

$$H_n(\sin\kappa s|q)\, \mathrm{e}^{-s^2/2} = \frac{\mathrm{i}^n}{\sqrt{2\pi}}\, q^{n^2/4} \int_{-\infty}^{\infty} h_n(\sinh\kappa r|q)\, \mathrm{e}^{-\mathrm{i}sr - r^2/2}\, \mathrm{d}r \qquad q = \exp(-2\kappa^2) \tag{8}$$

one can substitute (8) in the right-hand side of (7) and sum over n by the aid of the generating function

$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/2}}{(q;q)_n} h_n(\sinh\kappa r|q) = E_q(t\mathrm{e}^{\kappa r})\, E_q(-t\mathrm{e}^{-\kappa r}) \tag{9}$$

for the q^{-1}-Hermite polynomials $h_n(x|q)$ [13]. This yields (cf [9])

$$\mathcal{E}_q(\sin\kappa s;\alpha)\mathrm{e}^{-s^2/2} = \frac{e_{q^2}(q\alpha^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, E_q(\mathrm{i}\alpha q^{1/2}\mathrm{e}^{-\kappa r})\, E_q(-\mathrm{i}\alpha q^{1/2}\mathrm{e}^{\kappa r})\, \mathrm{d}r\,. \tag{10}$$

Observe that this integral representation may be considered as an analytic continuation of $\mathcal{E}_q(x;\alpha)$, initially defined by (6) for the values of $|\alpha| < 1$ only. Due to the factor $e_{q^2}(q\alpha^2)$, the q-exponential function $\mathcal{E}_q(x;\alpha)$ is meromorphic in the complex α plane with simple poles at $\alpha_n^{(\pm)} = \pm q^{-(n+1/2)}$, $n = 0, 1, 2, \ldots$ (cf [5]).

The q-analogues of the trigonometric functions $\cos\omega x$ and $\sin\omega x$ are respectively defined [10] as

$$C_q(x;\omega) = \tfrac{1}{2}[\mathcal{E}_q(x;\mathrm{i}\omega) + \mathcal{E}_q(x;-\mathrm{i}\omega)] \tag{11a}$$

$$S_q(x;\omega) = \frac{1}{2\mathrm{i}}[\mathcal{E}_q(x;\mathrm{i}\omega) - \mathcal{E}_q(x;-\mathrm{i}\omega)]\,. \tag{11b}$$

Combining equation (11) with the Fourier–Gauss transform (10) gives

$$C_q(\sin\kappa s;\omega)\,\mathrm{e}^{-s^2/2} = \frac{e_{q^2}(-q\omega^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_q(\omega q^{1/2}\mathrm{e}^{-\kappa r})\, E_q(-\omega q^{1/2}\mathrm{e}^{\kappa r})\, \mathrm{e}^{-r^2/2}\cos rs\, \mathrm{d}r \tag{12a}$$

$$S_q(\sin\kappa s;\omega)\,\mathrm{e}^{-s^2/2} = \frac{e_{q^2}(-q\omega^2)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_q(\omega q^{1/2}\mathrm{e}^{\kappa r})\, E_q(-\omega q^{1/2}\mathrm{e}^{-\kappa r})\, \mathrm{e}^{-r^2/2}\sin rs\, \mathrm{d}r\,. \tag{12b}$$

Another two integrals, involving the q-exponential function $\mathcal{E}_q(x;\alpha)$, follow from the Ramanujan-type orthogonality relation

$$\int_{-\infty}^{\infty} H_m(\sin\kappa s|q)\, H_n(\sin\kappa s|q)\, \mathrm{e}^{-s^2} \cos\kappa s\, \mathrm{d}s = \sqrt{\pi}\, q^{1/8}\, (q;q)_m\, \delta_{mn} \tag{13}$$

for the continuous q-Hermite polynomials [12]. Using the generating function (7) in (13) leads to

$$\int_{-\infty}^{\infty} H_m(\sin\kappa s|q)\, \mathcal{E}_q(\sin\kappa s;\alpha)\, \mathrm{e}^{-s^2} \cos\kappa s\, \mathrm{d}s = \sqrt{\pi}\, q^{\frac{1}{4}(m^2+\frac{1}{2})}\, \alpha^m\, e_{q^2}(q\alpha^2)\,. \tag{14}$$

Applying the same equation (7) to (14) gives

$$\int_{-\infty}^{\infty} \mathcal{E}_q(\sin\kappa s;\alpha)\, \mathcal{E}_q(\sin\kappa s;\beta)\, \mathrm{e}^{-s^2} \cos\kappa s\, \mathrm{d}s = \sqrt{\pi}\, q^{1/8}\, e_{q^2}(q\alpha^2)\, e_{q^2}(q\beta^2)\, E_q(q^{1/2}\alpha\beta)\,. \tag{15}$$

Relations (10), (12), (14) and (15) are consistent with the fact that the q-exponential function $\mathcal{E}_q(x;\alpha)$ is defined by (6) on the q-quadratic lattice. The key point in deriving these formulae is to represent the lattice as $x_q(s) = \sin\kappa s$, $q = \exp(-2\kappa^2)$.

In the case of the q-linear lattice $x_q(s) = \mathrm{e}^{\mathrm{i}\kappa s}$ the q-analogue of the exponential function e^z has the form [10]

$$\varepsilon_q(z) := \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{(q;q)_n} z^n . \tag{16}$$

The Fourier–Gauss transformation properties of this q-extension of the exponential function have been studied in [14]. The relations between (16) and the q-exponential functions $E_q(z)$ and $e_q(z)$ are

$$\varepsilon_q(\alpha \mathrm{e}^{\mathrm{i}\kappa s})\, \mathrm{e}^{-s^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, E_q(q^{1/4}\alpha \mathrm{e}^{\kappa r})\, \mathrm{d}r \tag{17}$$

$$\varepsilon_q(\alpha \mathrm{e}^{-\kappa s})\, \mathrm{e}^{-s^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, e_q(q^{-1/4}\alpha \mathrm{e}^{\mathrm{i}\kappa r})\, \mathrm{d}r . \tag{18}$$

Since $e_{1/q}(z) = E_q(-qz)$ and $\varepsilon_{1/q}(z) = \varepsilon_q(-qz)$, the Fourier–Gauss transforms (17) and (18) are interrelated by the substitution $q \to 1/q$ (i.e. $\kappa \to \mathrm{i}\kappa$).

Analogues of the trigonometric functions $\cos\omega x$ and $\sin\omega x$ on the q-linear lattice $c_q(\omega x)$ and $s_q(\omega x)$ [4, 10] are defined in terms of $\varepsilon_q(z)$ by the relations

$$c_q(\omega x) = \tfrac{1}{2}[\varepsilon_q(\mathrm{i}\omega x) + \varepsilon_q(-\mathrm{i}\omega x)] \tag{19a}$$

$$s_q(\omega x) = \frac{1}{2\mathrm{i}}[\varepsilon_q(\mathrm{i}\omega x) - \varepsilon_q(-\mathrm{i}\omega x)] \tag{19b}$$

respectively. Therefore, from equation (17) it follows that

$$c_q(\omega \mathrm{e}^{\mathrm{i}\kappa s})\, \mathrm{e}^{-s^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, Cos_q(q^{1/4}\omega \mathrm{e}^{\kappa r})\, \mathrm{d}r \tag{20a}$$

$$s_q(\omega \mathrm{e}^{\mathrm{i}\kappa s})\, \mathrm{e}^{-s^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, Sin_q(q^{1/4}\omega \mathrm{e}^{\kappa r})\, \mathrm{d}r . \tag{20b}$$

Here Jackson's q-trigonometric functions $Cos_q(z)$ and $Sin_q(z)$ are defined as [11]

$$Cos_q(z) = \tfrac{1}{2}[E_q(\mathrm{i}z) + E_q(-\mathrm{i}z)] \qquad Sin_q(z) = \frac{1}{2\mathrm{i}}[E_q(\mathrm{i}z) - E_q(-\mathrm{i}z)] . \tag{21}$$

In a like manner, for the linear lattice $x_{1/q}(s) = \mathrm{e}^{-\kappa s}$ from (18) we have

$$c_q(\omega \mathrm{e}^{-\kappa s})\, \mathrm{e}^{-s^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, \cos_q(q^{-1/4}\omega \mathrm{e}^{\mathrm{i}\kappa r})\, \mathrm{d}r \tag{22a}$$

$$s_q(\omega \mathrm{e}^{-\kappa s})\, \mathrm{e}^{-s^2/2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}sr - r^2/2}\, \sin_q(q^{-1/4}\omega \mathrm{e}^{\mathrm{i}\kappa r})\, \mathrm{d}r \tag{22b}$$

where [11]

$$\cos_q(z) = \tfrac{1}{2}[e_q(\mathrm{i}z) + e_q(-\mathrm{i}z)] \qquad \sin_q(z) = \frac{1}{2\mathrm{i}}[e_q(\mathrm{i}z) - e_q(-\mathrm{i}z)] . \tag{23}$$

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